Linear programming application in optimal control of an inventory “VANGUARD SCIENTIFIC INSTRUMENTS IN MANAGEMENT” (ISSN 1314-0582)

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Abstract – In this paper we consider the application of the Markovian decision processes for decision making in defining an optimal policy in order to control inventory systems. We assume that the demand process has a stochastic nature and can be described using a Poisson distribution. To define an optimal control policy, based on the Markovian decision processes we use a proper cost structure, sets of possible solutions and final number of states, described applying a proper transition matrix. In order to realize the optimal control policy we use the linear programming approach.

Keywords – optimization, inventory systems, decision-making, linear programming

JEL: C44 and C61

1. INTRODUCTION

The Markovian decision processes are a serious method to formulate models and to define optimal control strategies to a wide set of systems. Such a method is applied to solve problems in various domains like queuing and inventory models, production, service, transport and etc. *Taha (2006)* and *Gatev (2002)*. The theory of the Markovian decision processes studies the optimization of discrete-time stochastic systems. Each control policy, which is investigated, poses a stochastic process and values of the cost function associated with this process. One of the existing methods to define an optimal policy to control the inventory models is that of the *linear programming* (LP), see *Taha (2006)* and *Vanderbei (2001).*

Here we conduct optimal control of a particular inventory system. The aim of the paper is to present the application of the LP method to optimally control the considered inventory system. Further the manuscript is structured as follows. In the following Section we give theoretical aspects for the Markovian decision processes with LP. In Section 3 we perform the optimal control of the investigated inventory system applying the linear programming method. Finally in Section 4 we end up with some concluding remarks.

2. MARKOVIAN DECISION PROCESSES – THEORETICAL ASPECTS

The Markovian decision processes are defined as a stochastic system, in which the occurrence of a particular state depends on the previous state of the system. The stochastic process being a family of random elements is a Markov process with the following Markovian property:

$$P\{X_{t+1} = j / X_0 = K_0, X_1 = K_1, ... X_{t-1} = K_{t-1}, X_t = i\} = \\ = P\{X_{t+1} = j / X_t = i\} \quad (1)$$

for $t = 0, 1, 2, \cdots$ and for each combination of i , j , $K_0, K_1, ... K_{t-1}$.

The probability $P\{X_{t+1} = j / X_t = i\} = p_{ij}$ is considered to be a transition probability and expresses the conditional probability of the system to make a transition from state *i* to state *j*. This probability is called *one step transition probability.* With $p_{ij}^{(n)}$ we yield the *n* – step transition probability. It represents the conditional probability that the random element *X*, being in state *i* in moment *t*, moves to state *j* after exactly *n* steps. The expressions below are true for the conditional probabilities:

$$p_{ij}^{(n)} \geq 0, \ \sum_{j=0}^{M} p_{ij}^{(n)} = 1.$$

The transition probabilities can be given in a matrix form

State	*0*	*1*	*...*	*M*
0	$p_{00}^{(n)}$	$p_{01}^{(n)}$		$p_{0M}^{(n)}$
1	$p_{10}^{(n)}$	$p_{11}^{(n)}$		$p_{1M}^{(n)}$
⋮				
M	$p_{M0}^{(n)}$	$p_{M1}^{(n)}$		$p_{MM}^{(n)}$

A policy *R* is a rule for undertaking a decision for every period of time. It can use the whole information for the previous observations till *t.* This is the complete history of the system, containing the set of the states $X_0, X_1, ... X_t$ and the undertaken decisions $\Delta_0, \Delta_1, ... \Delta_{t-1}$. For most of the problems met in practice is enough to be accepted that the policy depends on the observed state in the moment *t*, X_t and the possible decisions. The policy *R* can be observed as a statement for undertaking a decision $d_i(R)$, in case the system is in state *i*, $i = \overline{1, M}$. A policy can be completely described using the elements $\{d_0(R), d_1(R), ... d_M(R)\}$.

An alternative to this representation is a policy described using the following matrix:

$$R = \begin{bmatrix} D_{01} & D_{02} & \cdots & D_{0K} \\ D_{11} & D_{12} & \cdots & D_{1K} \\ \vdots & & & \\ D_{M1} & D_{M2} & \cdots & D_{MK} \end{bmatrix}, \quad (2)$$

where *K* is the number of the possible solutions. In this representation in each row of the matrix *R* a single number one should exist and the rest of the elements should be zero. If the element $D_{iK} = 1$, then the interpretation is the following: solution *K* is undertaken, if the state of the system is *i*.

The expected costs can be represented as a function of the variables D_{ik}. The variables D_{ik} should have a value of zero or one, this means usage of bulean integer programming [4]. In the standard set up the linear programming problem requires continuous variables, which gives motivation an extended interpretation of term policy to be searched. Let the rule for undertaking a solution D_{ik} is defined using the conditional probability:

$$D_{ik} = P\{solution = K / state = i\}, \; i = \overline{1, M}, \; k = \overline{1, K} \quad (3)$$

Defined in this way the policy is called *randomized*, for a difference from the deterministic policy for which $D_{ik} = 1$ or $D_{ik} = 0$. The randomized policy is characterized with elements $0 \le D_{ik} \le 1$.

Assume that y_{ik} is the unconditional probability, that the system is in state *i* and solution *K* is undertaken.

$$y_{ik} = P\{solution = K \quad and \quad state = i\}. \quad (4)$$

This probability can be represented like this:

$$y_{ik} = P\{solution = K / state = i\}.P\{state = i\},$$

or

$$y_{ik} = D_{ik}\pi_i . \quad (5)$$

The following equality is valid:

$$\pi_i = \sum_{k=1}^{K} y_{ik} . \quad (6)$$

Therefore the solution D_{ik} can be represented using the linear programming problem variables, i.e. y_{ik}:

$$D_{ik} = \frac{y_{ik}}{\pi_i} = \frac{y_{ik}}{\sum_{k=1}^{K} y_{ik}}. \quad (7)$$

The linear programming problem constraints are the following: the sum of the steady-state probabilities is one $\sum_{i=0}^{M} \pi_i = 1$.

Also the following constraint follows has to be satisfied:

$$\sum_{i=0}^{M} \sum_{k=1}^{K} y_{ik} = 1 \quad (8)$$

from the approach to calculate the steady-state probabilities

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij} \ ,$$

the following constraint is obtained:

$$\sum_{k=1}^{K} y_{jk} = \sum_{i=0}^{M} \sum_{k=1}^{K} y_{ik} p_{ij}(k) \, . \quad (9)$$

The linear programming problem variables are the unconditional probabilities

$$y_{ik} \geq 0 \, , \ i = \overline{0, M} \, , \ k = \overline{1, K} \, .$$

The performance specification is the average steady-state expected cost per unit time:

$$E(C) = \sum_{i=0}^{M} \sum_{k=1}^{K} \pi_i C_{ik} D_{ik} = \sum_{i=0}^{M} \sum_{k=1}^{K} C_{ik} y_{ik} \ . \quad (10)$$

3. OPTIMAL INVENTORY CONTROL USING LP

We consider the linear programming method to optimally control a particular inventory system. There is a shop for electronic devices, which can be ordered weekly. Let with D_1, D_2, … we denote the demand of a certain TV set for the period one, period two and etc. We assume that D_i are independent and identically distributed random variables described using the Poisson distribution and are characterized with parameter λ. Let X_0 be the number of available TV devices in the beginning, and x_1- the number of TV sets at the end of the first period, x_2- the number of TV sets at the end of the second period and so on. At the end of each Saturday we make a delivery request, which can be accomplished on Monday early in the morning. We have to apply an optimal control policy to satisfy the inventory storage and the customers' demand. When the demand exceeds the available items in the storage then we have losses of not being able to satisfy the demand. The considered cost structure requires to have in mind penalty losses of *500 Leva* for each unit of not accomplished demand. If a number of *z* items are required, the corresponding costs amount to *50+250z Leva.* If items are not ordered then delivery costs are not taken into account. The storage costs are not considered in this case. We suppose that the maximum number of lots of TV sets available in the inventory is limited to four /4/.

In this manuscript we have to use the linear programming method aiming at computing an optimal control policy using the Markovian decision processes. The variable x_1 describes the state of the systems i.e., the number of initial lots at the end of the first period *t*, and the values which x_t can obtain are: x_t *=0, 1, 2, 3, 4*. The decisions which can be undertaken are four in Tab1.:

Table 1. Undertaken decisions and their description

Decision	*Action*
0	*No inquiry is made*
1	*Inquiry for 1item is made*
2	*Inquiry for 2 items*
3	*Inquiry for 3 items*
4	*Inquiry for 4 items*

Source: Own research

The sets of possible solutions vary depending on the states, the transition matrices for decisions 0, 1, 2, 3 and 4 are shown in Tab2., Tab.3, Tab.4, Tab.5 and Tab.6:

Table 2.Transition matrix for decision 0.

Decision 0					
State	*0*	*1*	*2*	*3*	*4*
0	*1*	*0*	*0*	*0*	*0*
1	$P\{D_t \geq 1\}$	$P\{D_t = 0\}$	*0*	*0*	*0*
2	$P\{D_t \geq 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*	*0*
3	$P\{D_t \geq 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*
4	$P\{D_t \geq 4\}$	$P\{D_t = 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$

Source: Own research

Table 3.Transition matrix for decision 1.

Desicion 1					
State	*0*	*1*	*2*	*3*	*4*
0	$P\{D_t \geq 1\}$	$P\{D_t = 0\}$	*0*	*0*	*0*
1	$P\{D_t \geq 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*	*0*
2	$P\{D_t \geq 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*
3	$P\{D_t \geq 4\}$	$P\{D_t = 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$
4	*Banned*				

Source: Own research

Table 4. .Transition matrix for decision 2.

Desicion 2					
State	*0*	*1*	*2*	*3*	*4*
0	$P\{D_t \geq 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*	*0*
1	$P\{D_t \geq 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*
2	$P\{D_t \geq 4\}$	$P\{D_t = 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$
3	*Banned*				
4	*Banned*				

Source: Own research

Table 5. Transition matrix for decision 3.

Desicion 3					
State	*0*	*1*	*2*	*3*	*4*
0	$P\{D_t \geq 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$	*0*
1	$P\{D_t \geq 4\}$	$P\{D_t = 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$
2	*Banned*				
3	*Banned*				
4	*Banned*				

Source: Own research

Table 6. Transition matrix for decision4.

Desicion 4					
State	*0*	*1*	*2*	*3*	*4*
0	$P\{D_t \geq 4\}$	$P\{D_t = 3\}$	$P\{D_t = 2\}$	$P\{D_t = 1\}$	$P\{D_t = 0\}$
1	*Banned*				
2	*Banned*				
3	*Banned*				
4	*Banned*				

Source: Own research

The expected costs for one period C_{ik} for each state for all of the decisions are presented in Tab. 7:

Table 7. Expected costs

i \ k	*0*	*1*	*2*	*3*	*4*
0	*521.05*	*445.7*	*417.4*	*461.75*	*556.35*
1	*278.17*	*292.4*	*336.75*	*431.35*	∞
2	*152.4*	*211.75*	*306.35*	∞	∞
3	*71.75*	*186.35*	∞	∞	∞
4	*41.35*	∞	∞	∞	∞

Source: Own research

The optimal control policy of the inventory system is based on the linear programming method formulation and we use the software package *MATLAB* and especially the function *linprog,*see *Vanderbei (2001)* and *Sierksma(2015),* to perform the computations. The formulas presented in the previous section are applied, which gives the opportunity to obtain the results below.

In the LP approach we do not compare different control policies at different steps here the optimal policy is computed directly using the considered optimization procedure.

The matrix of the unconditional probability is the following

$$y_{ik} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1859 \\ 0.0000 & 0.0000 & 0.0000 & 0.2020 & 0.0000 \\ 0.0000 & 0.0000 & 0.2707 & 0.0000 & 0.0000 \\ 0.2384 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1031 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

This means that the matrix representation of the deterministic optimal policy is of the following type

$$D_{ik} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

thus when the system is in states 0, 1 ,2 ,3, 4 we undertake decisions 4, 3, 2, 0, 0. The optimal value of the cost specification is 294.80 EURO.

5. CONCLUSIONS

In this manuscript we investigated the application of the Markovian decision processes in defining an optimal policy in order to control a concrete inventory system using the linear programming method. The obtained results are acceptable and can inspire our future investigations in order to optimally control more realistic inventory systems using the LP method.

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